DSP

Chapter-4 : Filter Realization

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Filter Design Process

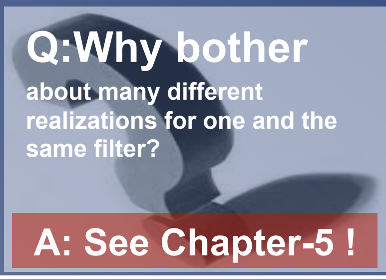
- **Step-1 : Define filter specs**
  Pass-band, stop-band, optimization criterion,…
- **Step-2 : Derive optimal transfer function**
  FIR or IIR design **Chapter-3**
- **Step-3 : Filter realization** (block scheme/flow graph)
  Direct form realizations, lattice realizations,… **Chapter-4**
- **Step-4 : Filter implementation** (software/hardware)
  Finite word-length issues, … **Chapter-5**
  Question: implemented filter = designed filter ?
  'You can't always get what you want' -Jagger/Richards (?)

Chapter-4 : Filter Realization

- **FIR Filter Realization**
- **IIR Filter Realization**

PS: Will always assume *real-valued* filter coefficients

PS:



FIR Filter Realization

FIR Filter Realization

=Construct (realize) LTI system (with delay elements, adders and multipliers), such that I/O behavior is given by..

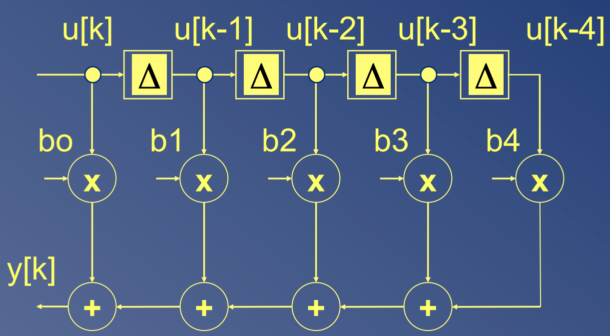
$$y[k] = b_0.u[k] + b_1.u[k-1] + \ldots + b_L.u[k-L]$$

Several possibilities exist…

1. Direct form
2. Transposed direct form
3. Lattice realization (LPC lattice)
4. Lossless lattice realization
5. Frequency-domain realization: see Part-IV

FIR / 1. Direct Form

$$y[k] = b_0.u[k] + b_1.u[k-1] + \ldots + b_L.u[k-L]$$

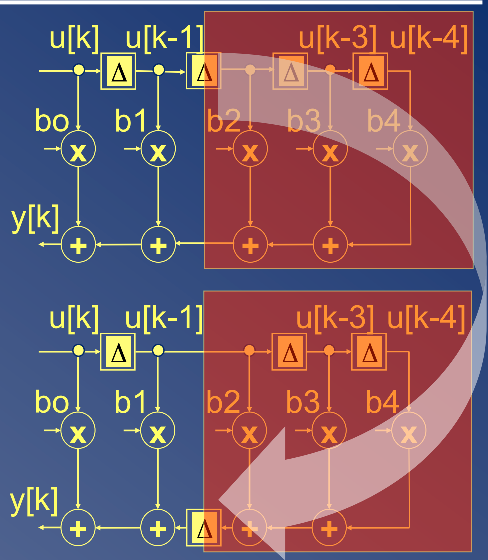


FIR / 2. Transposed Direct Form

Starting point is direct form :

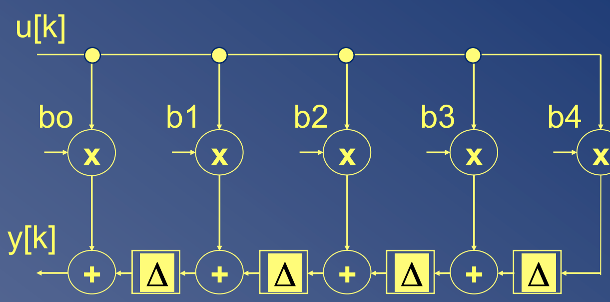
`Retiming' =

select subgraph (shaded)
remove one delay element on all inbound arrows add one delay element on all outbound arrows



FIR / 2. Transposed Direct Form

`Retiming' : repeated application results in...



i.e. `transposed direct form'

(=different software/hardware (`pipeline delays'), same i/o-behavior)

FIR / 3. Lattice Realization

Derived from combined realization of…

$$H(z): \qquad y[k] = b_0.u[k] + b_1.u[k-1] + \ldots + b_L.u[k-L]$$

…with `flipped' version of H(z)

$$\tilde{H}(z) = z^{-L}.H(z^{-1}): \quad \tilde{y}[k] = b_L.u[k] + b_{L-1}.u[k-1] + \ldots + b_0.u[k-L]$$

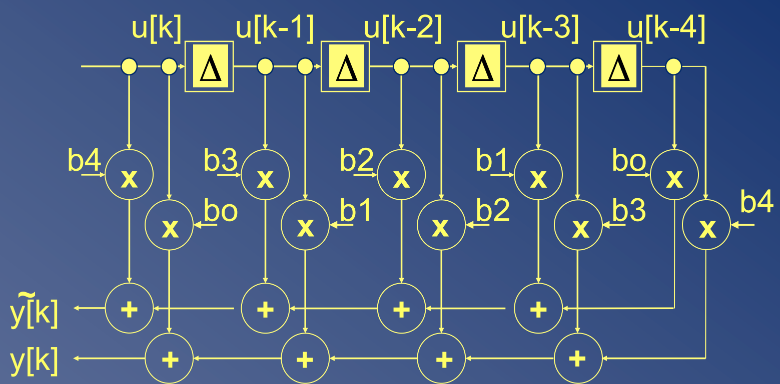
Reversed (real-valued) coefficient vector results in...

$$\left|\tilde{H}(z)\right|^2_{z=e^{j\omega}} = \tilde{H}(z).\tilde{H}(z^{-1})\Big|_{z=e^{j\omega}} = \ldots = \left|H(z)\right|^2_{z=e^{j\omega}}$$

i.e. - same magnitude response
- different phase response

FIR / 3. Lattice Realization

Starting point is direct form realization…



Now 1 page of maths…

FIR / 3. Lattice Realization

$$\begin{bmatrix}\tilde{Y}(z)\\ Y(z)\end{bmatrix} = \begin{bmatrix} b_4 & b_3 & b_2 & b_1 & b_0\\ b_0 & b_1 & b_2 & b_3 & b_4\end{bmatrix} . \begin{bmatrix}1 & z^{-1} & \dots & z^{-4}\end{bmatrix}^T . U(z)$$

With $\kappa_0 = \frac{b_4}{b_0}$ if $(b_0 \neq 0)$ and $(|\kappa_0| \neq 1)$ this can be rewritten as

$$\begin{bmatrix}\tilde{Y}(z)\\ Y(z)\end{bmatrix} = \begin{bmatrix}1 & \kappa_0\\ \kappa_0 & 1\end{bmatrix}\begin{bmatrix} \underline{\underline{0}} & b'_3 & b'_2 & b'_1 & b'_0\\ b'_0 & b'_1 & b'_2 & b'_3 & \underline{\underline{0}}\end{bmatrix} . \begin{bmatrix}1 & z^{-1} & \dots & z^{-4}\end{bmatrix}^T . U(z)$$

$$= \begin{bmatrix}1 & \kappa_0\\ \kappa_0 & 1\end{bmatrix} . \begin{bmatrix} z^{-1} & 0\\ 0 & 1\end{bmatrix} . \begin{bmatrix} b'_3 & b'_2 & b'_1 & b'_0\\ b'_0 & b'_1 & b'_2 & b'_3\end{bmatrix} . \begin{bmatrix}1 & z^{-1} & \dots & z^{-3}\end{bmatrix}^T . U(z)$$

order reduction

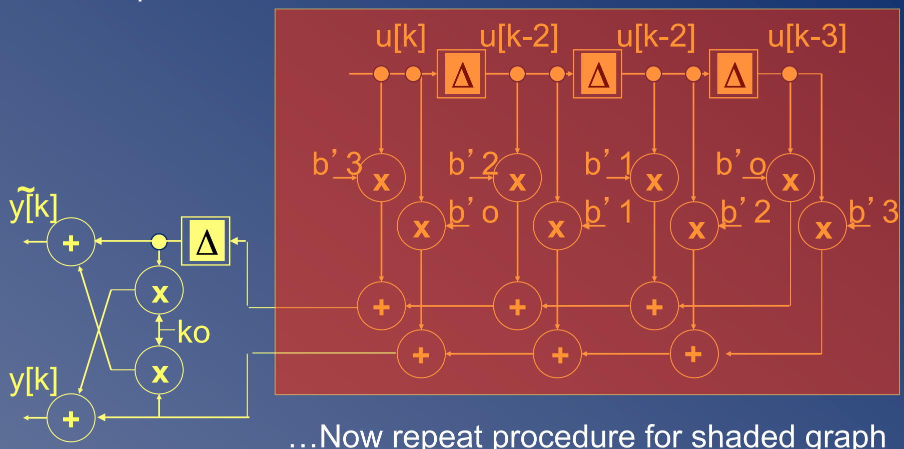
$$b'_0 = b_0, \quad b'_1 = \frac{b_1 - \kappa_0 b_3}{1-\kappa_0^2}, \quad b'_2 = \frac{b_2 - \kappa_0 b_2}{1-\kappa_0^2}, \quad b'_3 = \frac{b_3 - \kappa_0 b_1}{1-\kappa_0^2}$$

PS: find fix for case bo=0

PS: if |ko|=1, then transformation matrix $\begin{bmatrix}1 & \kappa_0\\ \kappa_0 & 1\end{bmatrix}$ is rank-deficient

FIR / 3. Lattice Realization

This is equivalent to...

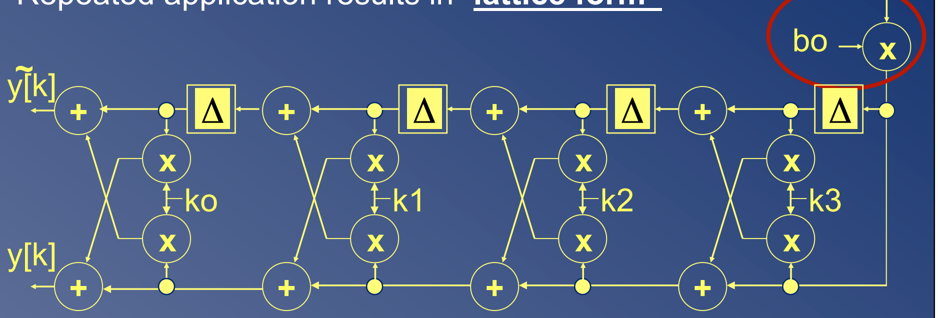


...Now repeat procedure for shaded graph

(=same structure as the one we started from)

FIR / 3. Lattice Realization

Repeated application results in `**lattice form'**



(= different software/hardware, same i/o-behavior)

FIR / 3. Lattice Realization

- Also known as `linear predictive coding (LPC) lattice '
  K_i's are so-called **`reflection coefficients'**
  Every set of b_i's corresponds to a set of K_i's, and vice versa
- Procedure for computing K_i's from b_i's corresponds to the well-known **`Schur-Cohn'** **stability test** (from control theory):
  Problem = for a given polynomial B(z), how do we find out if all the zeros of B(z) are 'stable' (i.e. lie inside unit circle) ?
  Solution = from bi's, compute reflection coefficients Ki's (=procedure on previous slides). Zeros are (proved to be) stable iff all Ki's statisfy |Ki|<1 !
- Procedure (page 10) breaks down if |Ki|=1 is encountered. Then have to select other realization (direct form, lossless lattice, …) for B(z)
- Lattice form not overly relevant at this point, but sets stage for similar derivations that lead to more relevant realizations (read on…)

FIR / 4. Lossless Lattice Realization

Derived from combined realization of H(z)…

$$H(z): \quad y[k] = b_0.u[k] + b_1.u[k-1] + \ldots + b_L.u[k-L]$$

…with

$$\tilde{\tilde{H}}(z): \quad \tilde{\tilde{y}}[k] = \tilde{\tilde{b}}_0.u[k] + \tilde{\tilde{b}}_1.u[k-1] + \ldots + \tilde{\tilde{b}}_L.u[k-L]$$

…which is such that

$$H(z).H(z^{-1}) + \tilde{\tilde{H}}(z).\tilde{\tilde{H}}(z^{-1}) = 1 \qquad (*)$$

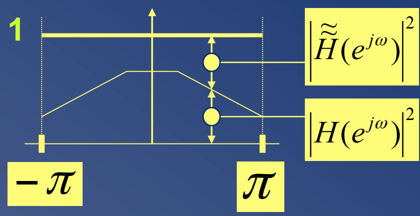
PS : Interpretation ?… (see next slide)

PS : May have to scale H(z) to achieve this (why?) (scaling omitted here)

FIR / 4. Lossless Lattice Realization

PS : Interpretation ?
When evaluated on the unit circle, formula (*) is equivalent to (for filters with real-valued coefficients)

$$\left|H(z)\right|^2_{z=e^{j\omega}} + \left|\tilde{\tilde{H}}(z)\right|^2_{z=e^{j\omega}} = 1$$



i.e. $H(z)$ and $\tilde{\tilde{H}}(z)$ are `power complementary'
(= form a 1-input/2-output `lossless' system, see also below)

FIR / 4. Lossless Lattice Realization

PS : How is $\tilde{\tilde{H}}(z)$ computed ?

$$\tilde{\tilde{H}}(z).\tilde{\tilde{H}}(z^{-1}) = \underbrace{1 - H(z).H(z^{-1})}_{R(z)}$$

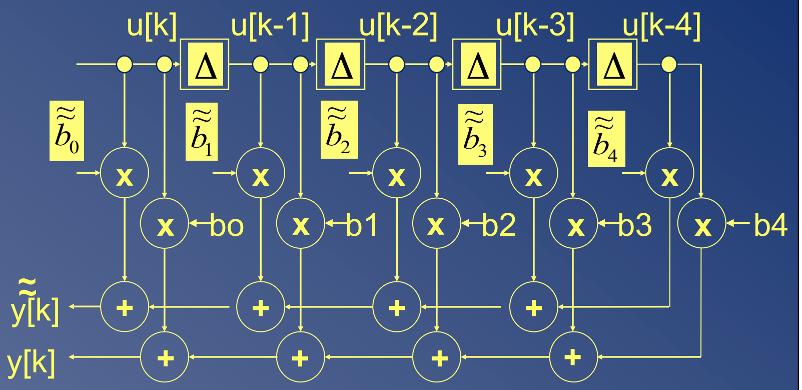
Note that if z_i (and z_i^*) is a root of R(z), then $1/z_i$ (and $1/z_i^*$) is also a root of R(z). Hence can factorize R(z) as…

$$\tilde{\tilde{H}}(z).\tilde{\tilde{H}}(z^{-1}) = \alpha^2 \prod_i (z^{-1} - z_i)(z - z_i) \rightarrow \tilde{\tilde{H}}(z) = \alpha \prod_i (z^{-1} - z_i)$$

Note that z_i's can be selected such that all roots of $\tilde{\tilde{H}}(z)$ lie inside the unit circle, i.e. $\tilde{\tilde{H}}(z)$ is a minimum-phase FIR filter.
This is referred to as spectral factorization, $\tilde{\tilde{H}}(z)$ =spectral factor.

FIR / 4. Lossless Lattice Realization

Starting point is direct form realization…



Now 1 page of maths…

FIR / 4. Lossless Lattice Realization

$$\begin{bmatrix} \tilde{\tilde{Y}}(z) \\ Y(z) \end{bmatrix} = \begin{bmatrix} \tilde{\tilde{b}}_0 & \tilde{\tilde{b}}_1 & \tilde{\tilde{b}}_2 & \tilde{\tilde{b}}_3 & \tilde{\tilde{b}}_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix} . \begin{bmatrix} 1 & z^{-1} & \dots & z^{-4} \end{bmatrix}^T . U(z)$$

From (*) (page 14), it follows that (**)

$$b_0.b_4 + \tilde{\tilde{b}}_0.\tilde{\tilde{b}}_4 = 0 \quad \Rightarrow \quad \begin{bmatrix} \tilde{\tilde{b}}_0 & b_0 \end{bmatrix} . \begin{bmatrix} \tilde{\tilde{b}}_4 \\ b_4 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} \tilde{\tilde{b}}_0 \\ b_0 \end{bmatrix} \perp \begin{bmatrix} \tilde{\tilde{b}}_4 \\ b_4 \end{bmatrix}$$

= orthogonal vectors

Hence there exists a rotation angle θo such that

$$\begin{bmatrix} \tilde{\tilde{Y}}(z) \\ Y(z) \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \underline{\underline{0}} & \tilde{\tilde{b}}'_0 & \tilde{\tilde{b}}'_1 & \tilde{\tilde{b}}'_2 & \tilde{\tilde{b}}'_3 \\ b'_0 & b'_1 & b'_2 & b'_3 & \underline{\underline{0}} \end{bmatrix} . \begin{bmatrix} 1 & z^{-1} & \dots & z^{-4} \end{bmatrix}^T . U(z)$$

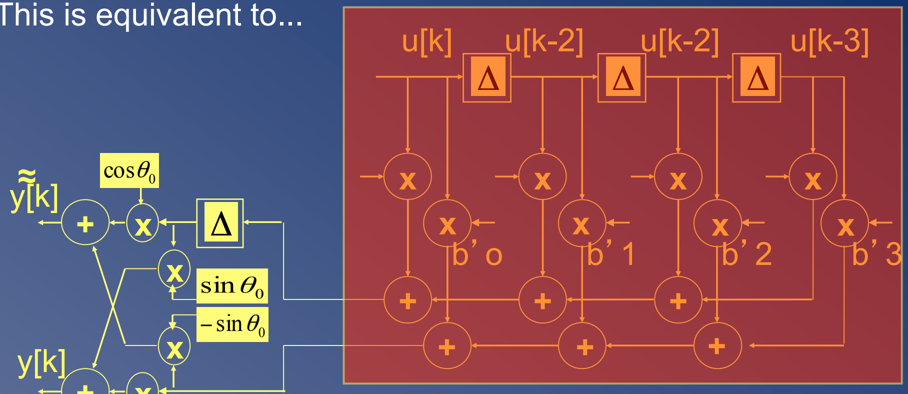
$$= \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} . \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} \tilde{\tilde{b}}'_0 & \tilde{\tilde{b}}'_1 & \tilde{\tilde{b}}'_2 & \tilde{\tilde{b}}'_3 \\ b'_0 & b'_1 & b'_2 & b'_3 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} & \dots & z^{-3} \end{bmatrix}^T . U(z)$$

order reduction

(**) $1 = H(z).H(z^{-1}) + \tilde{H}(z).\tilde{H}(z^{-1}) = \underbrace{(b_0.b_4 + \tilde{\tilde{b}}_0.\tilde{\tilde{b}}_4)}_{0}.(z^4 + z^{-4}) + \underbrace{(..)}_{0}.(z^3 + z^{-3}) + \dots + \underbrace{(..)}_{1}.z^0$

FIR / 4. Lossless Lattice Realization

This is equivalent to...



Now shaded graph can again be proved (***) to be power complementary system (Intuition? Hint: p.21).

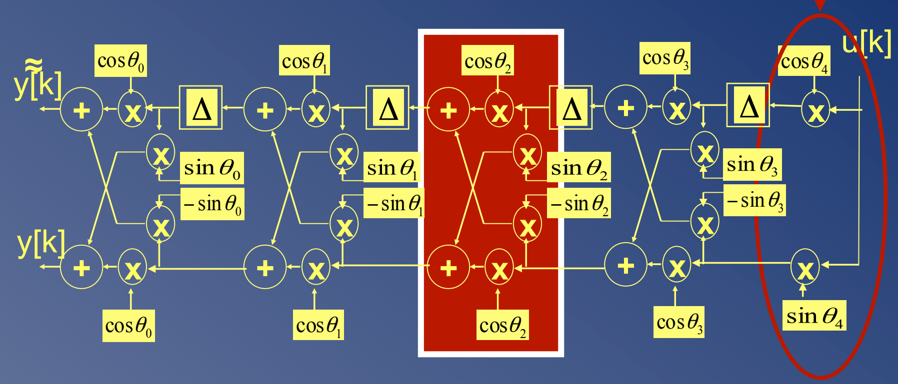
Hence can repeat procedure…

$$(***)\quad 1=\begin{bmatrix}\tilde{\tilde{H}}(z^{-1}) & H(z^{-1})\end{bmatrix}\cdot\begin{bmatrix}\tilde{\tilde{H}}(z)\\ H(z)\end{bmatrix}=\begin{bmatrix}\tilde{\tilde{H}}_{\otimes}(z^{-1}) & H_{\otimes}(z^{-1})\end{bmatrix}\cdot\begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\cos\theta_0 & \sin\theta_0\\ -\sin\theta_0 & \cos\theta_0\end{bmatrix}\cdot\begin{bmatrix}\cos\theta_0 & -\sin\theta_0\\ \sin\theta_0 & \cos\theta_0\end{bmatrix}\cdot\begin{bmatrix}z^{-1} & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\tilde{\tilde{H}}_{\otimes}(z)\\ H_{\otimes}(z)\end{bmatrix}=\begin{bmatrix}\tilde{\tilde{H}}_{\otimes}(z^{-1}) & H_{\otimes}(z^{-1})\end{bmatrix}\cdot\begin{bmatrix}\tilde{\tilde{H}}_{\otimes}(z)\\ H_{\otimes}(z)\end{bmatrix}$$

FIR / 4. Lossless Lattice Realization

Repeated application results in <u>**`lossless lattice'**</u>

explain



FIR / 4. Lossless Lattice Realization

Lossless lattice :

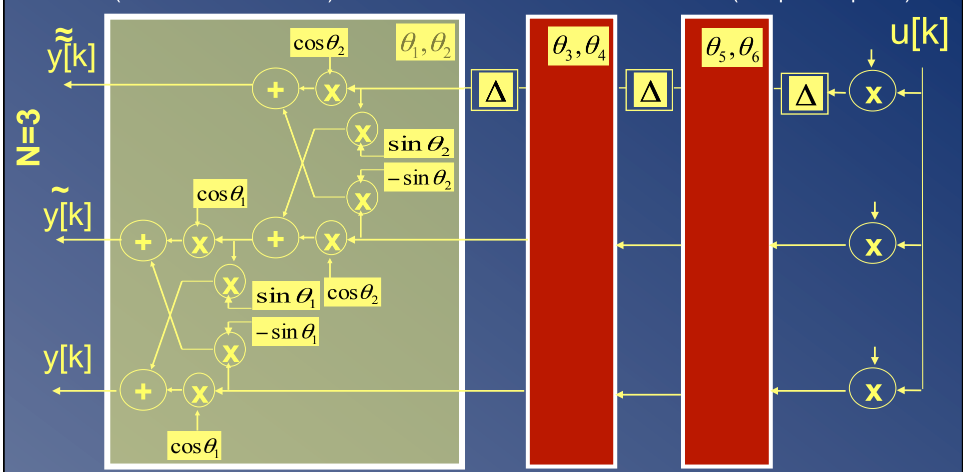
- Also known as `paraunitary lattice'
- Each **2-input/2-output section** is based on an orthogonal transformation, which preserves norm/energy/power

$$\begin{bmatrix} OUT_1 \\ OUT_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} IN_1 \\ IN_2 \end{bmatrix} \Rightarrow (IN_1)^2 + (IN_2)^2 = (OUT_1)^2 + (OUT_2)^2$$

  i.e. forms a 2-input/2-output `lossless' system (=time-domain view)
  Overall system is realized as cascade of lossless sections (+delays), hence is itself also `lossless' (see p.15, =freq-domain view)

FIR / 4. Lossless Lattice Realization

PS : Can be generalized to 1-input N-output lossless systems
(will be used in Part-IV) (compare to p.20 !)



$$H(z).H(z^{-1}) + \tilde{H}(z).\tilde{H}(z^{-1}) + \tilde{\tilde{H}}(z).\tilde{\tilde{H}}(z^{-1}) = 1$$

explain/derive!

IIR Filter Realization

IIR Filter Realization

=Construct (realize) LTI system (with delay elements, adders and multipliers), such that I/O behavior is given by..

$$H(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+...+b_Lz^{-L}}{1+a_1z^{-1}+...+a_Lz^{-L}}$$

Several possibilities exist…

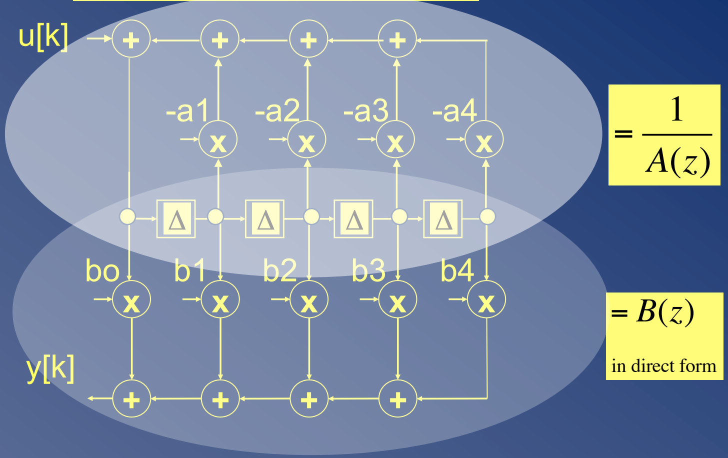
1. Direct form
2. Transposed direct form

PS: Parallel and cascade realization

4. Lattice-ladder realization
5. Lossless lattice realization

IIR / 1. Direct Form

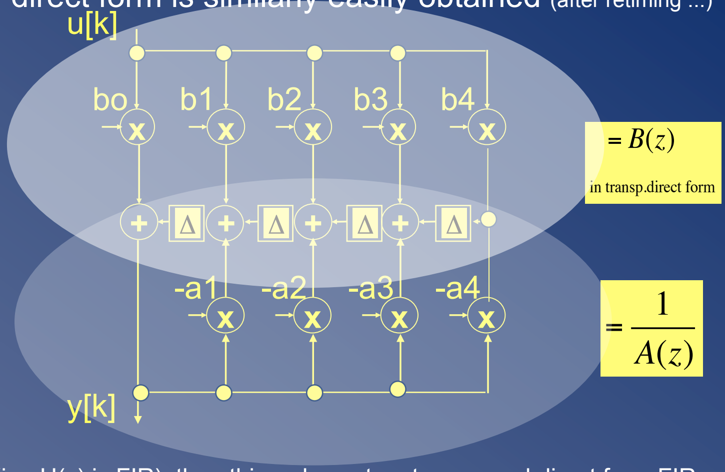
$$H(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+...+b_Lz^{-L}}{1+a_1z^{-1}+...+a_Lz^{-L}}$$



PS : If all ai=0 (i.e. H(z) is FIR), then this reduces to a direct form FIR

IIR / 2. Transposed Direct Form

Transposed direct form is similarly easily obtained (after retiming ...)



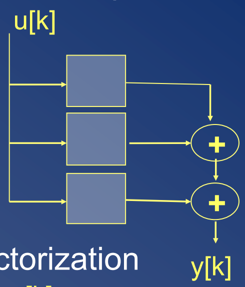
PS : If all a_i=0 (i.e. H(z) is FIR), then this reduces to a transposed direct form FIR

IIR / PS: Parallel & Cascade Realization

- Parallel realization based on partial fraction decomposition

For simple poles →

$$H(z)=\frac{B(z)}{A(z)}=c_0+\sum_{i=1}^{L_1}\frac{\alpha_i}{1+\beta_i z^{-1}}+\sum_{i=1}^{L_2}\frac{\gamma_1+\delta_i z^{-1}}{1+\varepsilon_1 z^{-1}+\varphi_N z^{-2}}$$

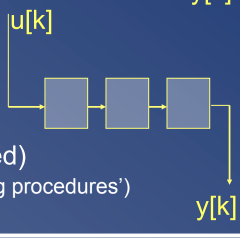


  - Each term realized in, e.g., direct form
  - Transmission zeros are realized iff signals from different sections exactly cancel out (=difficult in finite word-length implementation)

- Cascade realization based on pole-zero factorization

For L even →

$$H(z)=\frac{B(z)}{A(z)}=b_0\cdot\prod_{i=1}^{L/2}\frac{1+\alpha_i.z^{-1}+\beta_i.z^{-2}}{1+\gamma_i.z^{-1}+\delta_i.z^{-2}}$$



  - Each section('biquad') realized in, e.g., direct form
  - Cascade realization is not unique (details omitted) (=multiple ways of pairing poles and zeros (need for 'pairing procedures') and multiple ways of ordering sections in cascade)

IIR / 3. Lattice-Ladder Realization

Derived from combined realization of

$$H(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Lz^{-L}}{1+a_1z^{-1}+\ldots+a_Lz^{-L}}$$

with…

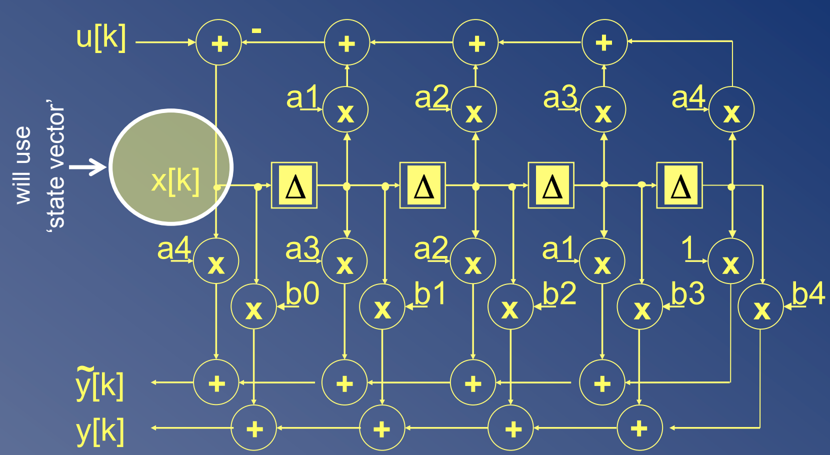
$$\tilde{H}(z)=\frac{\tilde{A}(z)}{A(z)}=\frac{a_L+a_{L-1}z^{-1}+\ldots+1.z^{-L}}{1+a_1z^{-1}+\ldots+a_Lz^{-L}}$$

- Numerator polynomial is denominator polynomial with reversed coefficient vector (see also p.8)
- Hence $\tilde{H}(z)$ is an `all-pass' (=`SISO lossless') filter :

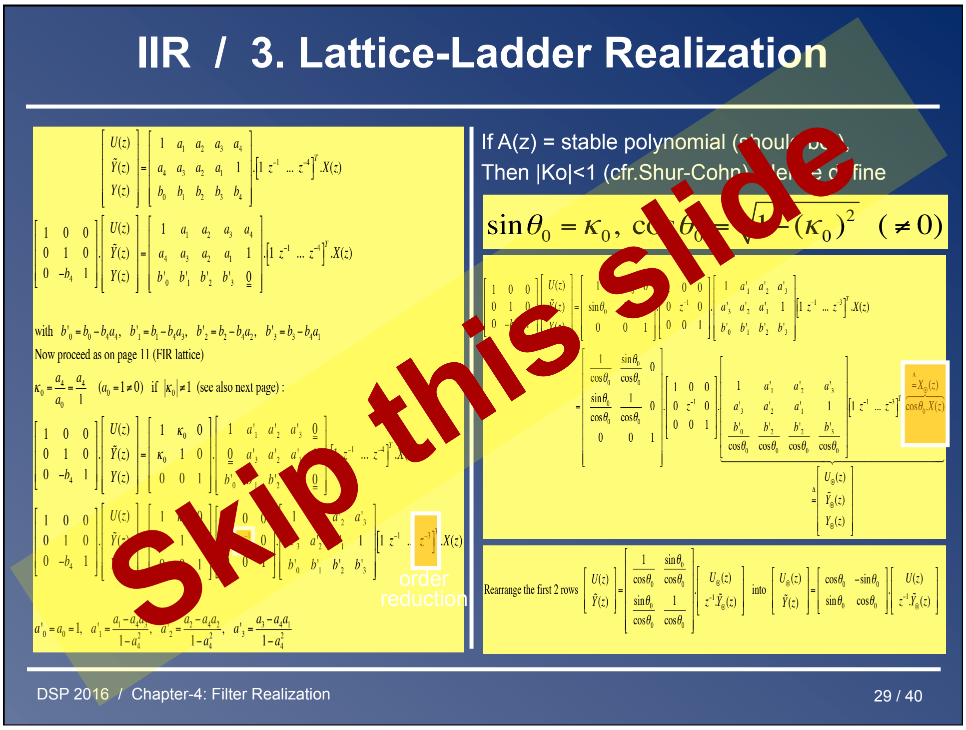
$$\tilde{H}(z).\tilde{H}(z^{-1})=1 \qquad \left|\tilde{H}(z)\right|^2_{z=e^{j\omega}}=\frac{\left|\tilde{A}(z)\right|^2_{z=e^{j\omega}}}{\left|A(z)\right|^2_{z=e^{j\omega}}}=1$$

IIR / 3. Lattice-Ladder Realization

Starting point is direct form realization…



Now 1 page (full) of maths…


IIR / 3. Lattice-Ladder Realization
Skip this slide
Then |Ko|<1 (cfr.Shur-Cohn)
define
with $b'_0=b_0-b_4a_4,\ b'_1=b_1-b_4a_3,\ b'_2=b_2-b_4a_2,\ b'_3=b_3-b_4a_1$
Now proceed as on page 11 (FIR lattice)
order reduction
Rearrange the first 2 rows
into
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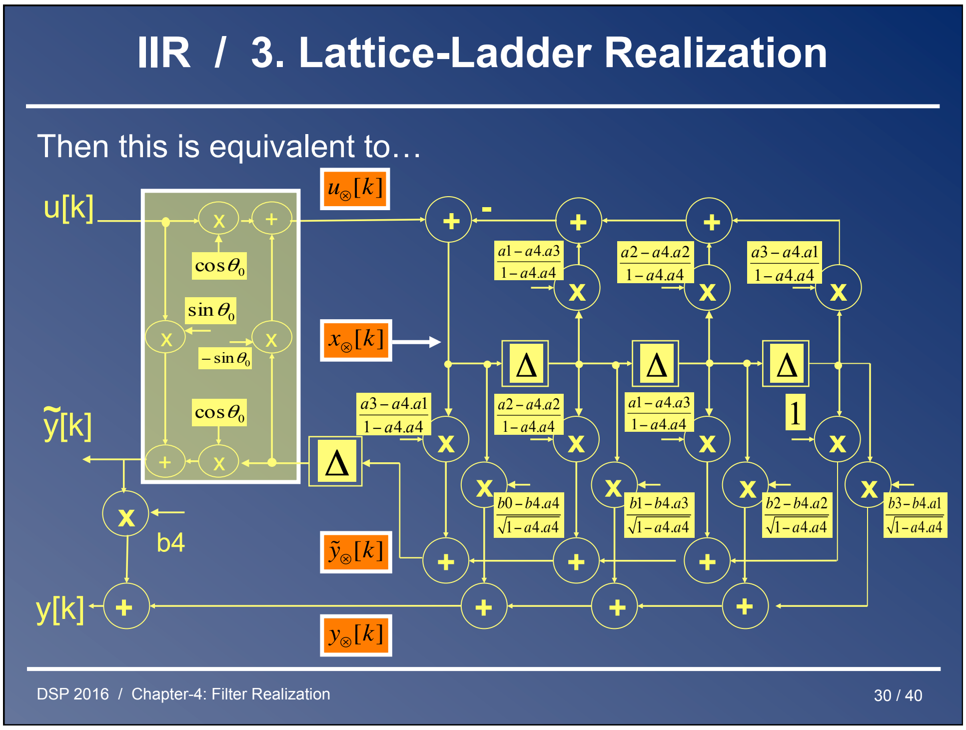
IIR / 3. Lattice-Ladder Realization
Then this is equivalent to…
u[k]
ỹ[k]
y[k]
b4
$\cos\theta_0$
$\sin\theta_0$
$-\sin\theta_0$
$u_{\otimes}[k]$
$x_{\otimes}[k]$
$\tilde{y}_{\otimes}[k]$
$y_{\otimes}[k]$
$\frac{a1-a4.a3}{1-a4.a4}$
$\frac{a2-a4.a2}{1-a4.a4}$
$\frac{a3-a4.a1}{1-a4.a4}$
$\frac{b0-b4.a4}{\sqrt{1-a4.a4}}$
$\frac{b1-b4.a3}{\sqrt{1-a4.a4}}$
$\frac{b2-b4.a2}{\sqrt{1-a4.a4}}$
$\frac{b3-b4.a1}{\sqrt{1-a4.a4}}$
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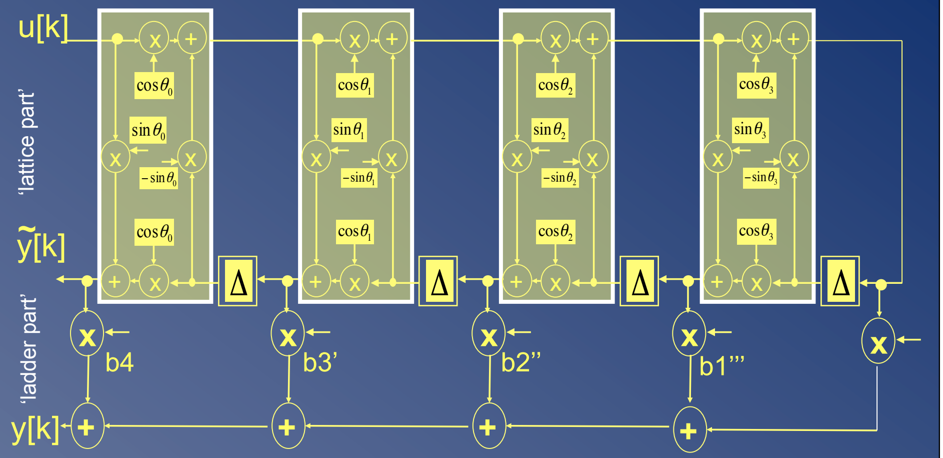
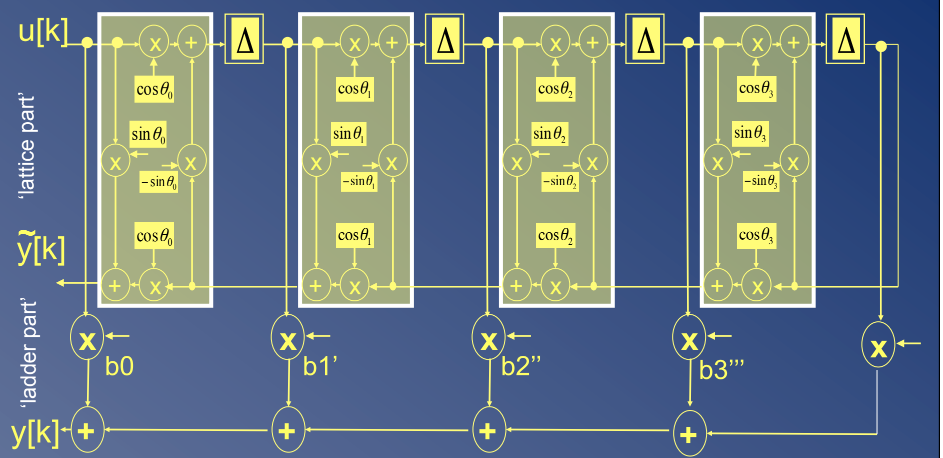
IIR / 3. Lattice-Ladder Realization

Procedure can be repeated (explain), leads to `lattice-ladder form'



IIR / 3. Lattice-Ladder Realization

PS: Similar derivation leads to 2^{nd} 'lattice-ladder' form



IIR / 3. Lattice-Ladder Realization

- K_i's **(=sin(theta$_i$) !)** are `reflection coefficients'
- Procedure for computing K_i's from a_i's again corresponds to **`Schur-Cohn' stability test**
- Orthogonal transformations correspond to 2-input/2-output `lossless' sections (=time-domain view).

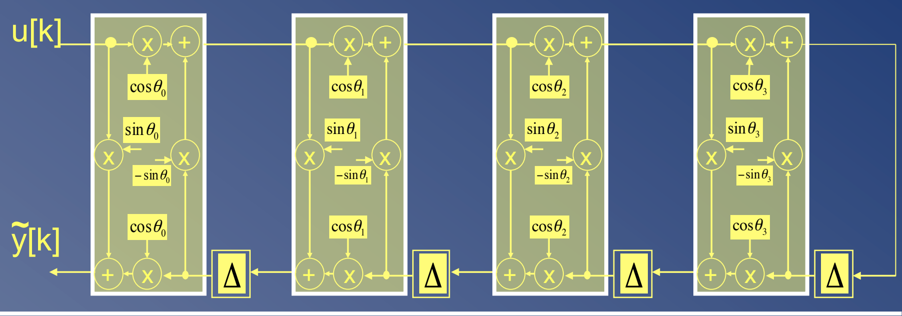
$$\begin{bmatrix} OUT_1 \\ OUT_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} IN_1 \\ IN_2 \end{bmatrix} \Rightarrow (IN_1)^2 + (IN_2)^2 = (OUT_1)^2 + (OUT_2)^2$$

Cascade of lossless sections (+delays) is also `lossless', i.e. `all-pass' (see p.27, =freq-domain view)

IIR / 3. Lattice-Ladder Realization

PS : Note that the all-pass part corresponds to A(z) (i.e. L angles θ_i correspond to L coeffs a_i) while the ladder part corresponds to B(z). If all a_i=0 (i.e. H(z) is FIR), then all θ_i=0, hence the all-pass part reduces to a delay line, and the lattice-ladder form reduces to a **direct-form FIR**.

PS : `All-pass' part (SISO u[k]->$\tilde{y}$[k]) is known as `**Gray-Markel**' structure



IIR / 4. Lossless Lattice Realization

Derived from combined realization of (possibly rescaled, as on p.14)

$$H(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Lz^{-L}}{1+a_1z^{-1}+\ldots+a_Lz^{-L}}$$

with...

$$\tilde{\tilde{H}}(z)=\frac{\tilde{\tilde{B}}(z)}{A(z)}=\frac{\tilde{\tilde{b}}_0+\tilde{\tilde{b}}_1.z^{-1}+\ldots+\tilde{\tilde{b}}_L.z^{-L}}{1+a_1z^{-1}+\ldots+a_Lz^{-L}}$$

such that…

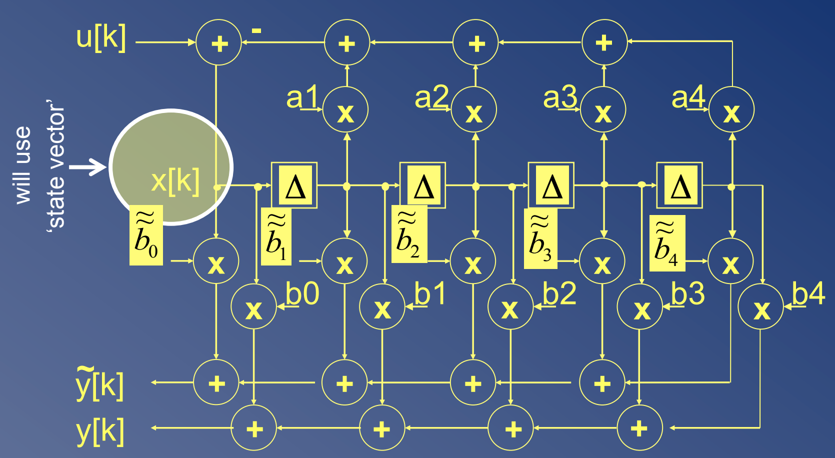
$$H(z).H(z^{-1})+\tilde{\tilde{H}}(z).\tilde{\tilde{H}}(z^{-1})=1 \qquad (**)$$

$$\Rightarrow B(z).B(z^{-1})+\tilde{\tilde{B}}(z).\tilde{\tilde{B}}(z^{-1})=A(z).A(z^{-1})$$

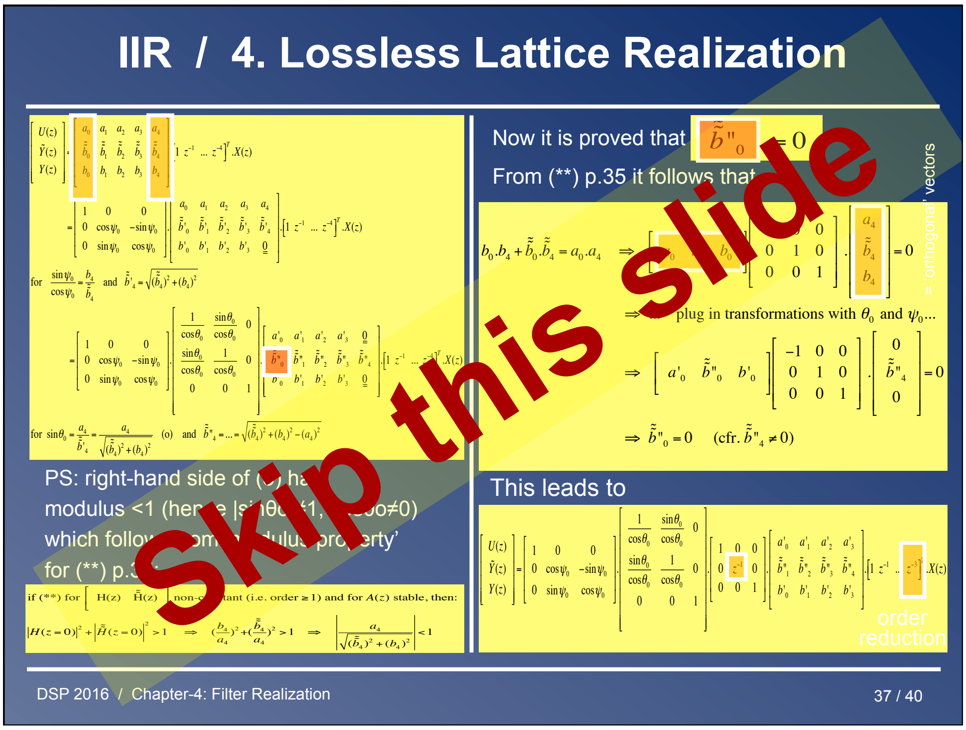
i.e. $\tilde{\tilde{H}}(z)$ and $H(z)$ are \`power complementary' (p.15)

IIR / 4. Lossless Lattice Realization

Starting point is direct form realization…



Now 1 page (full) of maths…


IIR / 4. Lossless Lattice Realization
Now it is proved that
From (**) p.35 it follows that
plug in transformations with θ_0 and ψ_0...
PS: right-hand side of
modulus <1
which follow
for (**) p.
This leads to
'orthogonal' vectors
order reduction
Skip this slide
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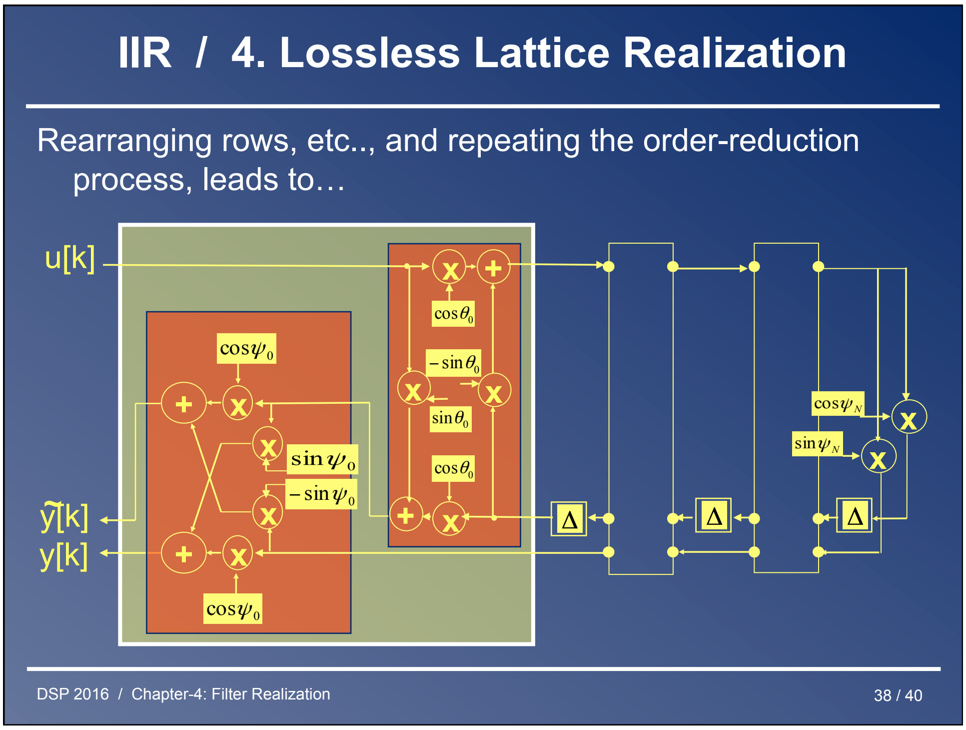
IIR / 4. Lossless Lattice Realization
Rearranging rows, etc.., and repeating the order-reduction process, leads to…
u[k]
ỹ[k]
y[k]
$\cos\psi_0$
$\sin\psi_0$
$-\sin\psi_0$
$\cos\theta_0$
$-\sin\theta_0$
$\sin\theta_0$
$\cos\psi_N$
$\sin\psi_N$
Δ
DSP 2016 / Chapter-4: Filter Realization
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IIR / 4. Lossless Lattice Realization

- Orthogonal transformations correspond to (3-input 3-output) \`lossless sections'

$$\begin{bmatrix} OUT_1 \\ OUT_2 \\ OUT_3 \end{bmatrix} = \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix} \begin{bmatrix} IN_1 \\ IN_2 \\ IN_3 \end{bmatrix}$$

$$\Rightarrow (IN_1)^2 + (IN_2)^2 + (IN_3)^2 = (OUT_1)^2 + (OUT_2)^2 + (OUT_3)^2$$

  Overall system is realized as cascade of lossless sections (+delays), hence is itself also \`lossless'

- PS : If all a_i=0 (i.e. H(z) is FIR), then all θ_i=0 and then this reduces to FIR lossless lattice !

- PS : If all φ_i=0, then this reduces to Gray-Markel structure !

IIR / 4. Lossless Lattice Realization

PS : Can be generalized to 1-input N-output lossless systems (=combine p.22 & p.38 !)